Exponentially Convergent Multiscale Methods Based on Edge Coupling: Example of Helmholtz Equation

Yixuan Wang*

Caltech
roywang@caltech.edu

*joint work with Thomas Hou, Yifan Chen

November 28, 2022

Section 1

Helmholtz Equation

Setting of Helmholtz Equation

Helmholtz equation with mixed boundary conditions:

$$\begin{cases} -\nabla \cdot (A\nabla u) - k^2 V^2 u = f, \text{ in } \Omega, \\ u = 0, \text{ on } \Gamma_D, \\ A\nabla u \cdot \nu = T_k u, \text{ on } \Gamma_N \cup \Gamma_R, \end{cases} \tag{1}$$

where $A_{\min} \leq A(x) \leq A_{\max}$, $\beta_{\min} \leq \beta(x) \leq \beta_{\max}$, $V_{\min} \leq V(x) \leq V_{\max}$, $T_k u = 0$ for $x \in \Gamma_N$, and $T_k u = ik\beta u$ for $x \in \Gamma_R$.

- Bilinear form:

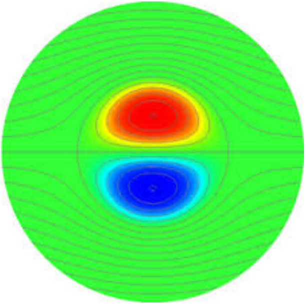
$$a(u, v) := (A\nabla u, \nabla v)_\Omega - k^2 (V^2 u, v)_\Omega - (T_k u, v)_{\Gamma_N \cup \Gamma_R}. \tag{2}$$

- Associated norm:

$$\|u\|_{\mathcal{H}(\Omega)} := \int_\Omega A|\nabla u|^2 + k^2 |Vu|^2. \tag{3}$$

Applications of Helmholtz Equation

1. Wave mechanics
2. Electrostatics
3. Seismology
4. Acoustics



Pollution Effect

I. Babuska, SINUM 1997.

- Mesh size sufficient to address the wave length: $O(1/k)$.
- For standard FEM: $h = O(1/k^2)$.
- Ideal method: $H = O(1/k)$!
  - $hp-$FEM with local polynomial of order $O(\log k)$. Melenk, Math. Comp., 2011.
  - Localizable orthogonal decompositions (LOD) with basis of support size $O(H \log(1/H))$. Peterseim, Math. Comp., 2014.
  - Multiscale edge basis with exponential rate of convergence.
  - A later work: Partition of unity method (PUM) with exponential rate of convergence. Ma, 2021.
- Fast solver with preconditioner: Ying, CPAM, 2011.

Sketch of Contributions

Our result: on a mesh of lengthscale $H = O(1/k)$, u can be computed by

$$u = \underbrace{\sum_{i \in I_1} c_i \psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i \in I_2} \psi_i^{(2)}}_{\text{(II)}} + C \exp(-b m^{\frac{1}{d+1}}) \qquad \text{(Energy norm)}$$

b, C constants independent of H, k. $\psi_i^{(1)}, \psi_i^{(2)}$ *local* support of size H.

- $\psi_i^{(1)}$ obtained by *local* SVD of $\mathcal{L}_\theta$ $\qquad \#I_1 = O(m/H^d)$
- $\psi_i^{(2)}$ obtained by solving *local* $\mathcal{L}_\theta u = f$ $\qquad \#I_2 = O(1/H^d)$
- c_i obtained by Galerkin's methods with basis functions $\psi_i^{(1)}$
- (II) $= O(H)$ (Energy norm)

A data-adaptive coarse-fine scale decomposition

Continuity Estimate and Stability

- Continuity estimate:

$$|a(u,v)| \leq C_c \|u\|_{\mathcal{H}(\Omega)} \|v\|_{\mathcal{H}(\Omega)}. \tag{4}$$

- Stability: Let $N_k f := u$ be the solution operator.

$$\sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|N_k f\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} =: C_{\text{stab}} < \infty. \tag{5}$$

Assumption on the stability constant: $C_{\text{stab}} \leq C_0 k^{\alpha}$.

Section 2

Coarse-Fine Scale Decomposition

Detour on Elliptic PDEs

- **Problem formulation**:

$$\begin{cases} -\nabla\cdot(a\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

$\Omega = [0,1]^2$ and $u \in H_0^1(\Omega), f \in L^2(\Omega)$.

- **Galerkin methods**: choose a finite-dim space $V_H \subset H_0^1(\Omega)$:

Find $u_H \in V_H$ such that $\int_\Omega a\nabla u_H \cdot \nabla v = \int_\Omega fv$ for any $v \in V_H$.

Optimality: (notation $\|u\|_{H_a^1(\Omega)} := \int_\Omega a|\nabla u|^2$)

$$\|u - u_H\|_{H_a^1(\Omega)} = \inf_{v\in V_H} \|u - v\|_{H_a^1(\Omega)}.$$

V_H needs to approximate the solution space well in the $H_a^1(\Omega)$ norm.

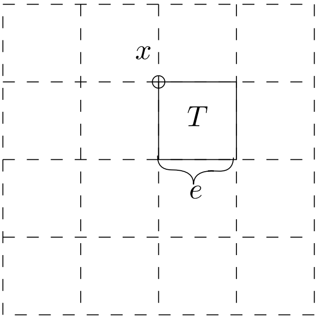
Explore the Solution Space

- **Mesh structure**:
  nodes, edges and elements.

- **Split** the solution locally:
  in each T, $u = u_T^{\mathsf{h}} + u_T^{\mathsf{b}}$.

$$\begin{cases} -\nabla\cdot(A\nabla u_T^{\mathsf{h}}) - k^2V^2u_T^{\mathsf{h}} = 0 & \text{in } T \\ u_T^{\mathsf{h}} = u & \text{on } \partial T, \end{cases}$$

$$\begin{cases} -\nabla\cdot(A\nabla u_T^{\mathsf{b}}) - k^2V^2u_T^{\mathsf{b}} = f & \text{in } T \\ u_T^{\mathsf{b}} = 0 & \text{on } \partial T. \end{cases}$$



$x \in \mathcal{N}_H, e \in \mathcal{E}_H, T \in \mathcal{T}_H$

- **Merge**: For each T, $u^{\mathsf{h}}(x) = u_T^{\mathsf{h}}(x)$ and $u^{\mathsf{b}}(x) = u_T^{\mathsf{b}}(x)$, when $x \in T$.

Coarse-fine Scale Decomposition

- **Poincaré inequality**: $\|v\|_{L^2(T)} \leq C_P H \|\nabla v\|_{L^2(T)}$.
- **Mesh assumption**: $H \leq A_{\min}^{1/2} / (\sqrt{2} C_P V_{\max} k)$.
- **Decomposition**: $u = u^{\mathsf{h}} + u^{\mathsf{b}} \in V^{\mathsf{h}} + V^{\mathsf{b}}$.

$$V^{\mathsf{h}} := \{v \in \mathcal{H}(\Omega) : -\nabla \cdot (A\nabla v) - k^2 V^2 v = 0 \text{ in each } T \in \mathcal{T}_H,$$
$$A\nabla v \cdot \nu = T_k v, \text{ on } \Gamma_N \cup \Gamma_R\} \quad (\textit{harmonic part})$$
$$V^{\mathsf{b}} := \{v \in \mathcal{H}(\Omega) : v = 0 \text{ on each } e \in \mathcal{E}_H\} \quad (\textit{bubble part})$$

For $v \in V^{\mathsf{h}}$ and $w \in V^{\mathsf{b}}$, it holds that $a(v, w) = 0$.

This decomposition makes sense by the C^α estimate of the solution.

Small Bubble Part

Bubble part is *local* and *small*:

- *local*: $u^{\mathsf{b}} = \sum_{i\in I_2^1} \psi_i^{(2)}$ (part of term (II)),
  each $\psi_i^{(2)}$ solves an elliptic equation inside each T.
- *small*: elliptic estimate,

$$\|u^{\mathsf{b}}\|_{\mathcal{H}(\Omega)} \leq \frac{3C_P}{A_{\min}^{1/2}} H \|f\|_{L^2(\Omega)}.$$

i.e. u^{b} oscillates at a frequency larger than $O(1/H)$.

Bubble part *is* the *fine scale* part.

Approximation of Harmonic Part

Observation: V^{h} is isomorphic to an edge space:

$$V^{\mathsf{h}} := \{v \in \mathcal{H}(\Omega) : -\nabla \cdot (A\nabla v) - k^2 V^2 v = 0 \text{ in each } T \in \mathcal{T}_H, \\ A\nabla v \cdot \nu = T_k v, \text{ on } \Gamma_N \cup \Gamma_R\}$$

Functions in V^{h}, locally solving Helmholtz-harmonic problems, only depend on values of v on edges.

Galerkin's solution u_H now *only* approximates the *harmonic* part.

Section 3

Exponentially Efficient Edge Basis

Localization to Edge Functions

- **Edge function**: $u^{\mathsf{h}} : \Omega \to \mathbb{R}$ restricted to edges: $\tilde{u}^{\mathsf{h}} : E_H \to \mathbb{R}$.

Task: find edge basis functions to approximate $\tilde{u}^{\mathsf{h}}$.

- **Localization to each edge**: $(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ vanishes at nodal points where I_H is nodal interpolation operator, e.g., by linear tent functions.

Next: find edge basis functions to approximate $(\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ for each e.

The edge residual $R_e \tilde{u}^{\mathsf{h}} := (\tilde{u}^{\mathsf{h}} - I_H \tilde{u}^{\mathsf{h}})|_e$ lies in the Lions-Magenes space, i.e. functions $v \in H^{1/2}(e)$ s.t. $\frac{v(x)}{\mathrm{dist}(x, \partial e)} \in L^2(e)$, by the C^α estimate.

Local Approximation via Oversampling

- **Oversampling**: $e \subset \omega_e := \overline{\bigcup\{T \in \mathcal{T}_H : \overline{T} \cap e \neq \emptyset\}}$.

$$\text{on } e: \ u^{\mathsf{h}} - I_H u^{\mathsf{h}} = (u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e}) + (u^{\mathsf{b}}_{\omega_e} - I_H u^{\mathsf{b}}_{\omega_e}).$$

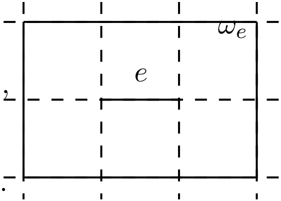
$u^{\mathsf{h}}_{\omega_e}, u^{\mathsf{b}}_{\omega_e}$: oversampling harmonic / bubble part.

- **Special harmonic function**: $u^{\mathsf{s}} \in V^{\mathsf{h}}$ is a special harmonic function such that its restriction on each edge $e \in E_H$ equals $\tilde{u}^{\mathsf{b}}_{\omega_e} - I_H \tilde{u}^{\mathsf{b}}_{\omega_e}$.

  Recall the definition:

$$\begin{cases} -\nabla \cdot (A\nabla u^{\mathsf{h}}_{\omega_e}) - k^2V^2 u^{\mathsf{h}}_{\omega_e} = 0 \ \text{ in } \omega_e \\ u^{\mathsf{h}}_{\omega_e} = u \ \text{ on } \partial\omega_e, \end{cases}$$

$$\begin{cases} -\nabla \cdot (A\nabla u^{\mathsf{b}}_{\omega_e}) - k^2V^2 u^{\mathsf{b}}_{\omega_e} = f \ \text{ in } \omega_e \\ u^{\mathsf{b}}_{\omega_e} = 0 \ \text{ on } \partial\omega_e. \end{cases}$$



Next: *Restrictions* of harmonic part are of *low* complexity!

Local Norm for Approximation

- **The $\mathcal{H}^{1/2}(e)$ norm**: (connect back to energy norms)

$$\|\tilde{\psi}\|^2_{\mathcal{H}^{1/2}(e)} := \int_\Omega A|\nabla\psi|^2 + k^2|V\psi|^2.$$

where ψ is the harmonic extension of $\tilde{\psi}$ to neighboring elements.

Theorem (Edge Coupling)

If on each edge, there is $\tilde{v}_e$ such that the local error satisfies

$$\|\tilde{u}^{\mathsf{h}}_{\omega_e} - I_H\tilde{u}^{\mathsf{h}}_{\omega_e} - \tilde{v}_e\|_{\mathcal{H}^{1/2}(e)} \le \epsilon_e,$$

then the global error satisfies

$$\|u^{\mathsf{h}} - u^{\mathsf{s}} - I_H u^{\mathsf{h}} - \sum_{e\in\mathcal{E}_H} v_e\|^2_{\mathcal{H}(\Omega)} \le C_{\text{mesh}} \sum_{e\in\mathcal{E}_H} \epsilon_e^2.$$

Low Complexity: Restrictions of Harmonic Part

Theorem (Y. Chen, T.Y. Hou, Y. Wang, 2021, 2022)

There exist constants C, b, such that for all m, we can find an $(m-1)$ dimensional space $W_e^m = \mathrm{span}\ \{\tilde{v}_e^k\}_{k=1}^{m-1}$ so that for any harmonic function v in ω_e,

$$\min_{\tilde{v}_e \in W_e^m} \|v - I_H v - \tilde{v}_e\|_{\mathcal{H}^{1/2}(e)} \leq C \exp\left(-b m^{\frac{1}{d+1}}\right) \|v\|_{\mathcal{H}(\omega_e)}.$$

- W_e^m obtained by left singular vectors of the operator $R_e v = v - I_H v$.
- Proof technique combines [Babuska, Lipton 2011] and C^α estimates.
- Essentially Helmholtz operator resembles an elliptic operator locally.

Summary of Approximations

- $u = u^{\mathsf{h}} + \overbrace{u^{\mathsf{b}}}^{\text{part of (II), small}}$ (harmonic-bubble splitting)

Summary of Approximations

- $u = u^{\mathsf{h}} + \overbrace{u^{\mathsf{b}}}^{\text{part of (II), small}}$ (harmonic-bubble splitting)
- $u^{\mathsf{h}} = \overbrace{(u^{\mathsf{h}} - I_H u^{\mathsf{h}})}^{\text{localized to each edge}} + \overbrace{I_H u^{\mathsf{h}}}^{\text{basis functions in (I)}}$ (interpolation part)

Summary of Approximations

- $u = u^{\mathsf{h}} + \overbrace{u^{\mathsf{b}}}^{\text{part of (II), small}}$ (harmonic-bubble splitting)
- $u^{\mathsf{h}} = \overbrace{(u^{\mathsf{h}} - I_H u^{\mathsf{h}})}^{\text{localized to each edge}} + \overbrace{I_H u^{\mathsf{h}}}^{\text{basis functions in (I)}}$ (interpolation part)
- $(u^{\mathsf{h}} - I_H u^{\mathsf{h}})|_e = \overbrace{(u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e})|_e}^{\text{restriction of harmonic part}} + \overbrace{u^{\mathsf{s}}|_e}^{\text{part of (II), small}}$

Summary of Approximations

- $u = u^{\mathsf{h}} + \overbrace{u^{\mathsf{b}}}^{\text{part of (II), small}}$ (harmonic-bubble splitting)
- $u^{\mathsf{h}} = \overbrace{(u^{\mathsf{h}} - I_H u^{\mathsf{h}})}^{\text{localized to each edge}} + \overbrace{I_H u^{\mathsf{h}}}^{\text{basis functions in (I)}}$ (interpolation part)
- $(u^{\mathsf{h}} - I_H u^{\mathsf{h}})|_e = \overbrace{(u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e})|_e}^{\text{restriction of harmonic part}} + \overbrace{u^{\mathsf{s}}|_e}^{\text{part of (II), small}}$
- $(u^{\mathsf{h}}_{\omega_e} - I_H u^{\mathsf{h}}_{\omega_e})|_e = \overbrace{\sum_{j=1}^{m-1} c_j v_e^j}^{\text{basis functions in (I)}} + C \exp\left(-bm^{\frac{1}{d+1}}\right) \|u^{\mathsf{h}}_{\omega_e}\|_{\mathcal{H}(\omega_e)}$

(I) basis functions not dependent on f, but on $\mathcal{L}_\theta$ (*local*)

(II) bubble part and special harmonic function (*local* and *small*)

Section 4

Multiscale Method

Overall Exponential Accuracy in Approximation

By the local to global error estimate, we have the overall approximation accuracy using $V_{H,m}$ consisting of basis functions in (I):

Theorem (Global Approximation)

$$\min_{v\in V_{H,m}} \|u^{\mathsf{h}} - u^{\mathsf{s}} - v\|_{\mathcal{H}(\Omega)} \leq C(C_{\mathrm{stab}}(k) + H)\exp\left(-bm^{\frac{1}{d+1}}\right)\|f\|_{L^2(\Omega)},$$

where C is a generic constant independent of k, m, H.

Multiscale Framework for Galerkin Methods

Handle coarse part $u^{\mathsf{h}} - u^{\mathsf{s}}$ and fine part $u^{\mathsf{b}} + u^{\mathsf{s}}$ separately.
Choose a finite-dim trial space $S \subset V^{\mathsf{h}}$, compute locally $u^{\mathsf{b}} + u^{\mathsf{s}}$, and then:

$$\text{find } u_S \in S \quad \text{such that} \quad a(u_S, v) = (f, v)_\Omega - a(u^{\mathsf{b}} + u^{\mathsf{s}}, v) \quad \text{for any } v \in S_{\text{test}}.$$

- $S_{\text{test}} = S$: Ritz-Galerkin;
- $S_{\text{test}} = \overline{S}$: Petrov-Galerkin.

Approximation Implies Accuracy

- **Approximation Ability**:

$$\eta^{\mathsf{h}}(S) := \sup_{f \in L^2(\Omega)\backslash\{0\}} \inf_{v \in S} \frac{\|u - v\|_{\mathcal{H}(\Omega)}}{\|f\|_{L^2(\Omega)}} \quad \text{with} \quad u = N_k f. \tag{6}$$

- Given that $k\eta^{\mathsf{h}}(S) \leq 1/(2C_c V_{\max})$, for the Ritz-Galerkin method with $\overline{S} = S$, we have **Quasi-optimal Approximation**:

$$\|u^{\mathsf{h}} - u^{\mathsf{s}} - u_S\|_{\mathcal{H}(\Omega)} \leq 2C_c \inf_{v \in S} \|u^{\mathsf{h}} - u^{\mathsf{s}} - v\|_{\mathcal{H}(\Omega)} .$$

Gårding-type inequality for a posteriori estimate.

Ritz-Galerkin Method

Theorem (Galerkin Exponential Accuracy)

Suppose $Ck[(C_{\text{stab}}(k) + H)\exp\left(-bm^{\frac{1}{d+1}}\right) + H] \leq 1/(2C_c V_{\max})$*, then using* $S = V_{H,m} + \overline{V_{H,m}}$ *in Ritz-Galerkin method leads to a solution* u_S *such that*

$$\|u^{\mathsf{h}} - u^{\mathsf{s}} - u_S\|_{\mathcal{H}(\Omega)} \leq 2C_c C(C_{\text{stab}}(k) + H)\exp\left(-bm^{\frac{1}{d+1}}\right)\|f\|_{L^2(\Omega)}.$$

- $m \sim \log^{d+2}(k)$ suffices for an exponential rate of convergence.
- $V_{H,m}$ and $\overline{V_{H,m}}$ only differ on the edges connected to the boundary, where Robin boundary conditions make the operator non-Hermitian.

Section 5

Numerical Experiments

High Wavenumber Example

- $A = V = \beta = 1$, $k = 2^7$, fine mesh $h = 2^{-10}$, coarse mesh $H = 2^{-5}$.
- Exact solution: $u(x_1, x_2) = \exp(-ik(0.6x_1 + 0.8x_2))$.

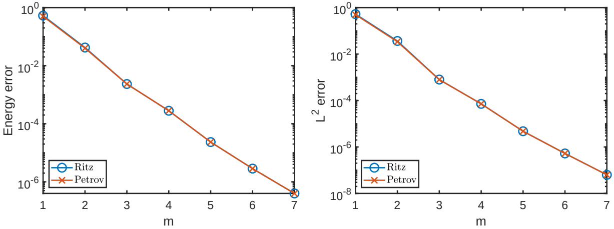


Figure: High wavenumber example. Left: $e_{\mathcal{H}}$ versus m; right: e_{L^2} versus m.

High Contrast Example: Mie resonances

$$\Omega_\varepsilon = (0.25, 0.75)^2 \cap \bigcup_{j\in\mathbb{Z}^2} \varepsilon\left(j + (0.25, 0.75)^2\right), \quad A(x) = \begin{cases} 1, & x \notin \Omega_\varepsilon \\ \varepsilon^2, & x \in \Omega_\varepsilon\,. \end{cases}$$

$\beta = 1, V = 1, k = 9.$

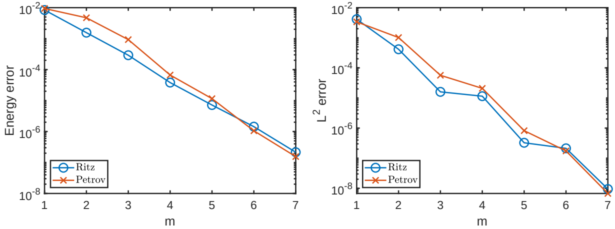
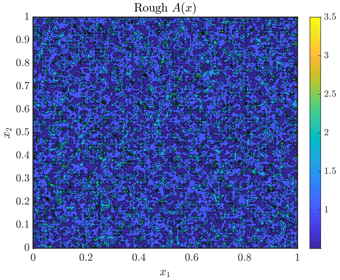


Figure: High contrast example. Left: $e_{\mathcal{H}}$ versus m; right: e_{L^2} versus m.

Mixed Boundary and Rough Field Example

Rough media with mixed boundary conditions. (Artificial)



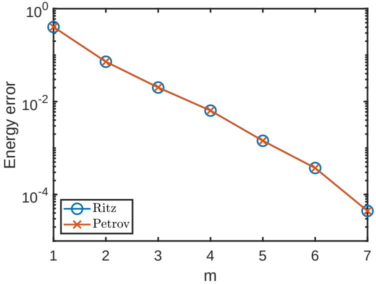


Figure: Left: the contour of A; right: relative errors in the energy norm.

Section 6

Conclusions

Summary of the Framework

1. Galerkin solution as a quasi-optimal approximation.
2. Harmonic-bubble decomposition to avoid non positive definiteness in the whole domain.
3. Local nodal/edge basis construction for global error estimate.
4. Exponential decay of the error by oversampling method to achieve optimal design.
5. Extensive numerical experiments to corroborate the exponential rate of convergence.

Sketch of Contributions

Our result: on a mesh of lengthscale $H = O(1/k)$, u can be computed by

$$u = \underbrace{\sum_{i \in I_1} c_i \psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i \in I_2} \psi_i^{(2)}}_{\text{(II)}} + C \exp(-bm^{\frac{1}{d+1}}) \qquad \text{(Energy norm)}$$

b, C constants independent of H, k. $\psi_i^{(1)}, \psi_i^{(2)}$ *local* support of size H.

- $\psi_i^{(1)}$ obtained by *local* SVD of $\mathcal{L}_\theta$ $\qquad \#I_1 = O(m/H^d)$
- $\psi_i^{(2)}$ obtained by solving *local* $\mathcal{L}_\theta u = f$ $\qquad \#I_2 = O(1/H^d)$
- c_i obtained by Galerkin's methods with basis functions $\psi_i^{(1)}$
- (II) $= O(H)$ (Energy norm)
- (I) Galerkin basis are fully offline.

A data-adaptive coarse-fine scale decomposition

Future Work

1. Generalization to other non-elliptic (time-dependent) problems, e.g. the Schrödinger equation, where the non-elliptic term could be treated as a perturbation term.
2. Generalization to higher-order operators and higher-dimensions.

References

Y. Chen, T. Y. Hou, and Y. Wang. *Exponentially convergent multiscale methods for high frequency heterogeneous Helmholtz equations*. 2021. arXiv: 2105.04080 [math.NA].

Yifan Chen, Thomas Y Hou, and Yixuan Wang. "Exponential convergence for multiscale linear elliptic PDEs via adaptive edge basis functions". In: *Multiscale Modeling & Simulation* 19.2 (2021), pp. 980–1010.

Thanks!