Exponentially Convergent Multiscale Finite Element Method

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Multiscale Problem

$$\begin{cases} -\nabla\cdot(A\nabla u) + Vu = f, \text{ in } \Omega \\ u = 0, \text{ on } \Gamma_1 \\ A\nabla u \cdot \nu = \beta u, \text{ on } \Gamma_2 . \end{cases}$$

- A, V, β are functions in $L^\infty(\Omega)$ and can be rough. $\partial\Omega = \Gamma_1 \cup \Gamma_2$
- $V = 0$: elliptic; $V = -k^2$: Helmholtz. Highly oscillatory solutions.
- Multiscale modeling aims at construct offline basis for multi-queries.
- Weak formulation via bilinear form $a(u, v)$; energy norm $\|\cdot\|_{\mathcal{H}}$.

Our Contributions

Our result [3]: on a mesh of length scale H, u can be computed by

$$u = \underbrace{\sum_{i\in I_1} c_i\psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i\in I_2} \psi_i^{(2)}}_{\text{(II)}} + C\exp(-bm^{\frac{1}{d+1}}) \quad \text{(Energy norm)}$$

b, C: constants independent of H. $\psi_i^{(1)}, \psi_i^{(2)}$: **local** support of size H.

- $\psi_i^{(1)}$ obtained by **local** SVD of $\mathcal{L}_\theta$ $\quad \#I_1 = O(m/H^d)$
- $\psi_i^{(2)}$ obtained by solving **local** $\mathcal{L}_\theta u = f$ $\quad \#I_2 = O(1/H^d)$
- c_i obtained by Galerkin's methods with basis functions $\psi_i^{(1)}$
- (II) $= O(H)$ (Energy norm)

A data-adaptive coarse-fine scale decomposition

For the Helmholtz equation [2], we require $H = O(1/k)$ whereas traditional FEM needs $H = O(1/k^2)$. The constants are independent of k as well.

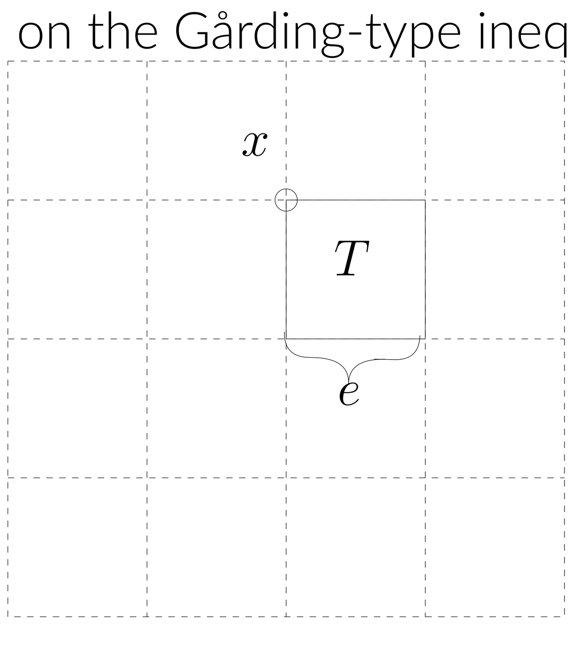
Approximation and Solution Space

Key observation: Galerkin solution $\to$ best approximation in energy norm in the subspace for elliptic problems with Dirichlet zero boundary condition. Similar quasi-optimality in approximation holds based on the Gårding-type inequality.

Split the solution locally:
in each T, $u = u_T^{\mathsf{h}} + u_T^{\mathsf{b}}$.

$$\begin{cases} -\nabla\cdot(A\nabla u_T^{\mathsf{h}}) + Vu_T^{\mathsf{h}} = 0, \text{ in } T \\ u_T^{\mathsf{h}} = u, \text{ on } \partial T\setminus\Gamma_2 \\ A\nabla u_T^{\mathsf{h}}\cdot\nu = \beta u_T^{\mathsf{h}}, \text{ on } \partial T\cap\Gamma_2 , \end{cases}$$

$$\begin{cases} -\nabla\cdot(A\nabla u_T^{\mathsf{b}}) + Vu_T^{\mathsf{b}} = f, \text{ in } T \\ u_T^{\mathsf{b}} = 0, \text{ on } \partial T\setminus\Gamma_2 \\ A\nabla u_T^{\mathsf{b}}\cdot\nu = \beta u_T^{\mathsf{b}}, \text{ on } \partial T\cap\Gamma_2 . \end{cases}$$



$x \in \mathcal{N}_H, e \in \mathcal{E}_H, T \in \mathcal{T}_H$

Merge: For each T, $u^{\mathsf{h}}(x) = u_T^{\mathsf{h}}(x)$ and $u^{\mathsf{b}}(x) = u_T^{\mathsf{b}}(x)$, when $x \in T$.

Locally decomposed approximation

Goal: approximate u^{h}, which corresponds to edge values

Edge Localization

- Harmonic extension Q_{E_H}: maps the edge values $\tilde{u}^{\mathsf{h}} = u^{\mathsf{h}}|_{E_H}$ to $u^{\mathsf{h}} \in H^1(\Omega)$.
- **Nodal interpolation**:

$$I_H\tilde{u} = \sum_{x_i\in\mathcal{N}_H} \tilde{u}(x_i)\tilde{\psi}_i$$

$\tilde{\psi}_i$: linear tent functions on E_H and we obtain nodal basis as $Q_{E_H}\tilde{\psi}_i$.

- **Edge restriction**: localized edge functions supported on each edge e.

$$R_e u := (\tilde{u} - I_H\tilde{u})|_e$$

- Local to global estimate:

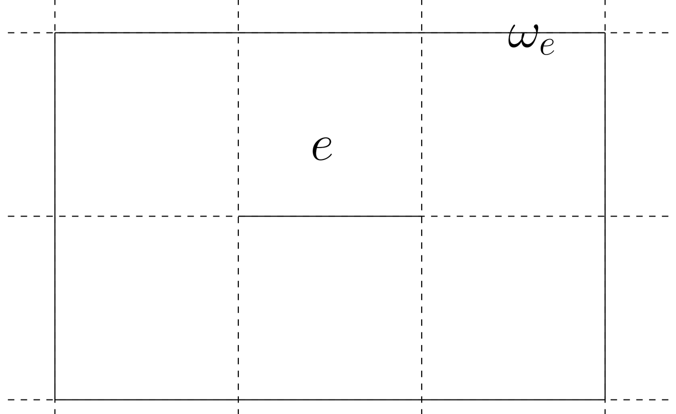
$$\|Q_{E_H}R_e u - w_e\|_{\mathcal{H}(\Omega)} \le \epsilon_e ,$$

then the global approximation error satisfies

$$\|Q_{E_H}(\tilde{u} - I_H\tilde{u}) - \sum_{e\in\mathcal{E}_H} w_e\|^2_{\mathcal{H}(\Omega)} \le C_{\text{mesh}} \sum_{e\in\mathcal{E}_H} \epsilon_e^2 ,$$

Edge Basis via Oversampling

Edge functions on e?
Oversampling information of
harmonic functions in patch ω_e!



$$Q_{E_H}R_e u = \underbrace{Q_{E_H}R_e}_{\text{SVD}} u^{\mathsf{h}}_{\omega_e} + \underbrace{Q_{E_H}R_e u^{\mathsf{b}}_{\omega_e}}_{\text{local}} .$$

SVD of the operator $Q_{E_H}R_e$ from oversampling harmonic functions to edge functions is exponentially convergent: edge basis $\{v_{e,j}\}_{j=1,\cdots,m}$.

$$\|Q_{E_H}R_e u^{\mathsf{h}}_{\omega_e} - \sum_{1\le j\le m} b_{e,j}v_{e,j}\|_{\mathcal{H}(\Omega)} \le C\exp(-bm^{\frac{1}{d+1}})\|u\|_{\mathcal{H}(\omega_e)} .$$

Multiscale Solver with Exponential Accuracy

$$u = u^{\mathsf{h}} + u^{\mathsf{b}} = \sum_{e\in\mathcal{E}_H}\sum_{1\le j\le m} b_{e,j}v_{e,j} + \sum_{x_i\in\mathcal{N}_H} u(x_i)\psi_i + u^{\mathsf{n}} + O(\exp(-bm^{\frac{1}{d+1}})\|u\|_{\mathcal{H}(\Omega)}) .$$

The final approximation, with $u^{\mathsf{n}} := u^{\mathsf{b}} + \sum_{e\in\mathcal{E}_H} Q_{E_H}R_e u^{\mathsf{b}}_{\omega_e}$ the local online part.

Effective equation for $u - u^{\mathsf{n}}$:

$$a(u - u^{\mathsf{n}}, v) = (f, v)_\Omega - a(u^{\mathsf{n}}, v) ,$$

1. We collect Galerkin basis **offline**: edge basis from local SVD and nodal basis.
2. Everytime for a new right-hand side, we compute local **online** part u^{n}.
3. We solve the effective equation for $u - u^{\mathsf{n}}$ via Galerkin method.
4. Exponential accuracy of combined solution by quasi-optimal approximation.

Numerical experiments

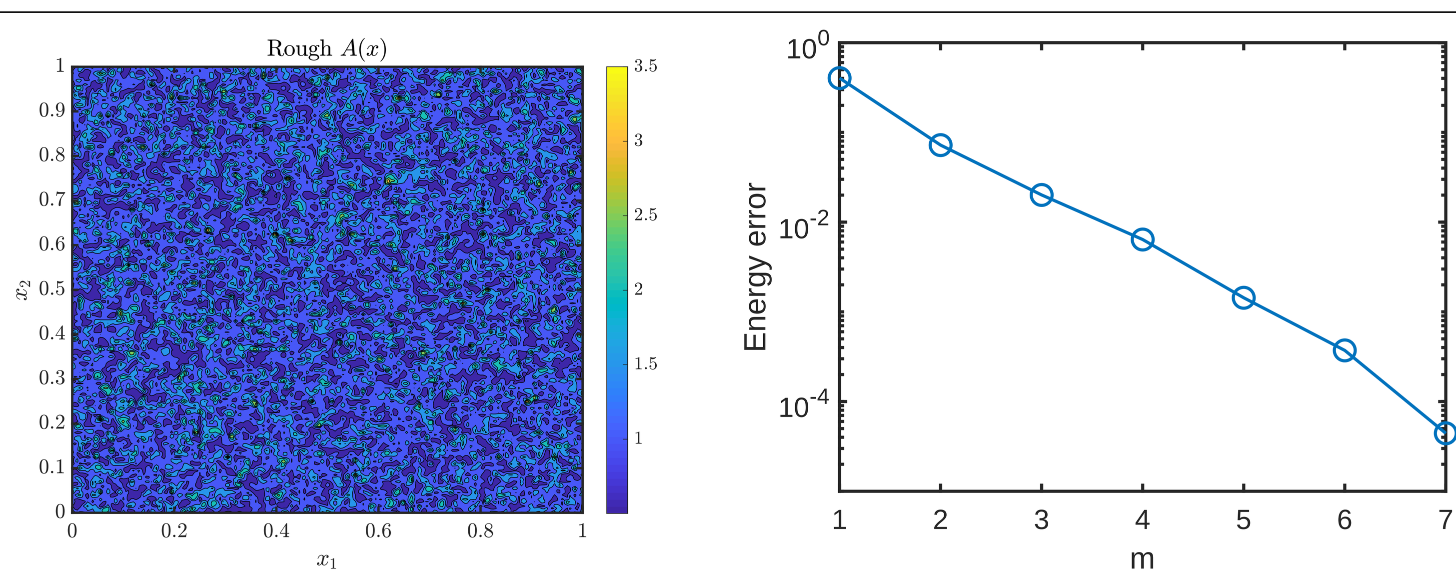


Figure 1. Left: the contour of A; right: relative error in energy norm versus m.

Example: Helmholtz equation with rough coefficients, mixed boundary condition and high wavelength $k = 2^5$. Coarse mesh $H = 2^{-5}$

Generalizations

Generalization to 3D problems

Nodal, edge and face basis. Exponential efficiency is established theoretically.

Generalization to time dependent problems

Schrödinger equations using ExpMsFEM in space coupled with appropriate time discretizations; also other time dependent problems.

Data driven model reduction

Operator learning to solve local problems efficiently.

References

[1] Yifan Chen, Thomas Y Hou, and Yixuan Wang.
Exponential convergence for multiscale linear elliptic pdes via adaptive edge basis functions.
Multiscale Modeling & Simulation, 19(2):980–1010, 2021.

[2] Yifan Chen, Thomas Y Hou, and Yixuan Wang.
Exponentially convergent multiscale methods for high frequency heterogeneous helmholtz equations.
arXiv preprint arXiv:2105.04080, 2021.

[3] Yifan Chen, Thomas Y Hou, and Yixuan Wang.
Exponentially convergent multiscale finite element method.
arXiv preprint arXiv:2212.00823, 2022.