Approximation to Singular Quadratic Collision Model in Fokker-Planck-Landau Equation

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Section 1

FPL Equation

Boltzmann and FPL Equation

Boltzmann equation:

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (vf) = \int (f(v_1')f(v') - f(v_1)f(v))B(|g|, \chi)d\chi dn dv_1 \qquad (1)$$

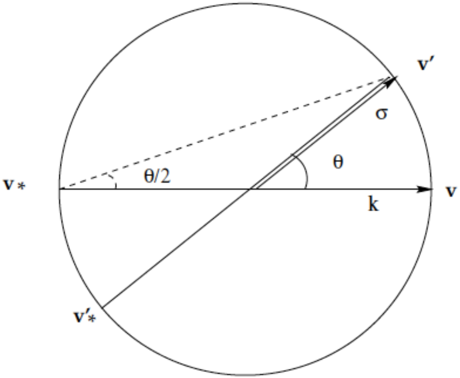
Fokker-Planck-Landau equation:

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f) = \mathcal{Q}[f], \qquad t \in \mathbb{R}^+, \quad \boldsymbol{x} \in \mathbb{R}^3, \quad \boldsymbol{v} \in \mathbb{R}^3, \qquad (2)$$

$$\mathcal{Q}[f](t, \boldsymbol{x}, \boldsymbol{v}) = \nabla_{\boldsymbol{v}} \cdot \int_{\mathbb{R}^3} A(\boldsymbol{v} - \boldsymbol{v}_*)(f(\boldsymbol{v}_*)\nabla_{\boldsymbol{v}} f(\boldsymbol{v}) - f(\boldsymbol{v})\nabla_{\boldsymbol{v}_*} f(\boldsymbol{v}_*))\,\mathrm{d}\boldsymbol{v}_*, \qquad (3)$$

where $A(\boldsymbol{v}) = \Psi(|\boldsymbol{v}|)\Pi(\boldsymbol{v})$, $\Pi_{ij}(\boldsymbol{v}) = \delta_{ij} - \frac{v_i v_j}{|\boldsymbol{v}|^2}$, $\Psi(|\boldsymbol{v}|) = \Lambda|\boldsymbol{v}|^{\gamma+2}$.

FPL Equation as a Limit Case



When all collision becomes grazing, i.e, the collision angle tends to zero.

Parameter γ

Different choice of γ leads to different models.

1. $\gamma > 0$: “hard potential”
2. $\gamma < 0$: “soft potential”
3. $\gamma = 0$: “Maxwell molecules”
4. $\gamma = -3$: Coulombian case

Domain of definition: $\gamma > -5$. Small γ leads to singularity.

Numerical Methods

1. Monte-Carlo Simulation
2. Discrete Velocity Methods
3. Fourier Spectral Methods

Section 2

Hermite Spectral Method

Linear FP and Hermite Diagonalization

For the Maxwell molecules with $\Lambda = 1$, if the distribution function f is radially symmetric, which is to be preserved under time evolution,

$$\mathcal{Q}^{\text{linear}}[f] = 2\nabla_{\boldsymbol{v}} \cdot (\nabla f + f\boldsymbol{v}), \tag{4}$$

$$\mathcal{Q}^{\text{linear}}[f] = \sum_{|\alpha|=0}^{+\infty} Q_{\alpha}^{\text{linear}} H^{\alpha}(\boldsymbol{v})\mathcal{M}(\boldsymbol{v}), \qquad Q_{\alpha}^{\text{linear}} = -(D-1)|\alpha| f_{\alpha}. \tag{5}$$

Local Maxwellian and Frame of Reference

Steady state solution, homogenous case:

$$f(\infty, \boldsymbol{v}) = \mathcal{M}_{\rho,\boldsymbol{u},\theta}(\boldsymbol{v}) := \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{u}|^2}{2\theta}\right), \tag{6}$$

where the density ρ, velocity $\boldsymbol{u}$ and temperature θ can be obtained by

$$\rho = \int_{\mathbb{R}^3} f(t,\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}, \quad \boldsymbol{u} = \frac{1}{\rho}\int_{\mathbb{R}^3} \boldsymbol{v} f(t,\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}, \quad \theta = \frac{1}{3\rho}\int_{\mathbb{R}^3} |\boldsymbol{v}-\boldsymbol{u}|^2 f(t,\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}.$$

Non-dimensionalization

$$\rho = 1, \quad \boldsymbol{u} = 0, \quad \theta = 1, \tag{7}$$

Series Expansion

Series expansion in the weighted L^2 space $\mathcal{F} = L^2(\mathbb{R}^3; \mathcal{M}^{-1}\,\mathrm{d}\boldsymbol{v})$:

$$f(t, \boldsymbol{v}) = \sum_{|\alpha|=0}^{+\infty} f_\alpha(t) H^\alpha(\boldsymbol{v}) \mathcal{M}(\boldsymbol{v}), \tag{8}$$

Hermite polynomials:

$$H^\alpha(\boldsymbol{v}) = \frac{(-1)^n}{\mathcal{M}(\boldsymbol{v})} \frac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1} \partial v_2^{\alpha_2} \partial v_3^{\alpha_3}} \mathcal{M}(\boldsymbol{v}), \tag{9}$$

Moments

The relationship between the coefficients f_α and the moments can be derived from the orthogonality of Hermite polynomials

$$\int_{\mathbb{R}^3} H^\alpha(\boldsymbol{v}) H^\beta(\boldsymbol{v}) \mathcal{M}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{v} = \delta_{\alpha,\beta} \alpha!, \tag{10}$$

Therefore,

$$f_\alpha = \frac{1}{\alpha!} \int H^\alpha(\boldsymbol{v}) f(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{v}, \tag{11}$$

Section 3

Approximation of Quadratic Collision Term

Series Expansion of the Collisional Term

$$\mathcal{Q}[f](\boldsymbol{v}) = \sum_{|\alpha|=0}^{+\infty} Q_\alpha H^\alpha(\boldsymbol{v})\mathcal{M}(\boldsymbol{v}). \tag{12}$$

$$Q_\alpha = \frac{1}{\alpha!}\int H^\alpha(\boldsymbol{v})\mathcal{Q}[f](\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = \sum_{|\lambda|=0}^{+\infty}\sum_{|\kappa|=0}^{+\infty} A_\alpha^{\lambda,\kappa} f_\lambda f_\kappa, \tag{13}$$

$$A_\alpha^{\lambda,\kappa} = \frac{1}{\alpha!}\int_{\mathbb{R}^3} H^\alpha(\boldsymbol{v})\nabla_{\boldsymbol{v}}\cdot\int_{\mathbb{R}^3}\mathrm{d}\boldsymbol{v}_*\,\mathrm{d}\boldsymbol{v} A(\boldsymbol{v}-\boldsymbol{v}_*)$$
$$\Big(H^\lambda(\boldsymbol{v}_*)\mathcal{M}(\boldsymbol{v}_*)\nabla_{\boldsymbol{v}}\left(H^\kappa(\boldsymbol{v})\mathcal{M}(\boldsymbol{v})\right) - H^\lambda(\boldsymbol{v})\mathcal{M}(\boldsymbol{v})\nabla_{\boldsymbol{v}_*}\left(H^\kappa(\boldsymbol{v}_*)\mathcal{M}(\boldsymbol{v}_*)\right)\Big).$$

Variational Perspective

Galerkin spectral method:

$$\mathcal{F}_M = \text{span}\{H^\alpha(v)M(v)|\alpha \in I_M\} \subset \mathcal{F} = L^2(\mathbb{R}^3; \mathcal{M}^{-1}dv), \tag{14}$$

where $I_M = \{(\alpha_1, \alpha_2, \alpha_3)|0 \leqslant |\alpha| \leqslant M, \alpha_i \in \mathbb{N}, i = 1, 2, 3\}$. Then the semi-discrete discrete function $f_M(t, \cdot) \in \mathcal{F}_M$ satisfies

$$\int_{\mathbb{R}^3} \frac{\partial f_M}{\partial t} \varphi \mathcal{M}^{-1} \,\mathrm{d}\boldsymbol{v} = \int_{\mathbb{R}^3} \mathcal{Q}(f_M, f_M) \varphi \mathcal{M}^{-1} \,\mathrm{d}\boldsymbol{v}, \qquad \forall \varphi \in \mathcal{F}_M. \tag{15}$$

Suppose

$$f_M(t, \boldsymbol{v}) = \sum_{\alpha \in I_M} f_\alpha(t) H^\alpha(\boldsymbol{v}) \mathcal{M}(\boldsymbol{v}) \in \mathcal{F}_M. \tag{16}$$

Model Reduction for the Quadratic Operator

The variational form (15) is equivalent to the following ODE system:

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = \sum_{\lambda \in I_M} \sum_{\kappa \in I_M} A_\alpha^{\lambda,\kappa} f_\lambda f_\kappa, \qquad \alpha \in I_M. \tag{17}$$

To reduce the time and storage cost, the coefficients $A_\alpha^{\lambda,\kappa}$ for a small number M_0 are computed and stored. When $\alpha \notin I_M$, we apply the linear model,

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = -(D-1)|\alpha| f_\alpha, \qquad \alpha \notin I_M. \tag{18}$$

Combining (17) and (18), we actually get a new collision operator

$$\mathcal{Q}^M[f] = P_M \mathcal{Q}[P_M f] - \mathcal{Q}^{\text{linear}}[(I - P_M)f], \qquad \forall f \in \mathcal{F}, \tag{19}$$

where P_M is the orthogonal projection from $\mathcal{F}$ onto $\mathcal{F}_M$.

The Novel Model

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = \mathcal{Q}_\alpha^M, \tag{20}$$

where

$$\mathcal{Q}_\alpha^M = \begin{cases} \sum_{\lambda \in I_{M_0}} \sum_{\kappa \in I_{M_0}} A_\alpha^{\lambda,\kappa} f_\lambda f_\kappa, & \alpha \in I_{M_0}, \\ -(D-1)|\alpha| f_\alpha, & \text{otherwise.} \end{cases} \tag{21}$$

Simplified models with higher accuracy can be chosen, for example, the linearized Boltzmann operator

Now we need to simplify the expression of the coefficients $A_\alpha^{\lambda,\kappa}$.

Theorem

$$A_\alpha^{\lambda,\kappa} = 2^{(\gamma+3-|\alpha|)/2} \sum_{s,t=1}^{3} \sum_{|p|=0}^{|\alpha|-1} \frac{\Lambda}{q^{[s]}!} \left(a_{p,r^{[t]}}^{\kappa+e_t,\lambda} - a_{p,r^{[t]}}^{\lambda,\kappa+e_t} \right) B_{r^{[t]}}^{q^{[s]}}(\gamma, s, t), \tag{22}$$

where $p = (p_1, p_2, p_3)^T$ *is a three-dimensional multi-index and*

$$q^{[s]} = \alpha - e_s - p, \quad r^{[t]} = \lambda + \kappa + e_t - p, \quad a_{p,q}^{\lambda,\kappa} = \prod_{i=1}^{3} a_{p_i q_i}^{\lambda_i \kappa_i}, \qquad s,t = 1,2,3. \tag{23}$$

The coefficients $a_{pq}^{\lambda\kappa}$ and $B_p^q(\gamma, s, t)$ are defined by

$$a_{pq}^{\lambda\kappa} = 2^{-(p+q)/2}\lambda!\kappa! \sum_{s=\max(0,p-\kappa)}^{\min(p,\lambda)} \frac{(-1)^{q-\lambda+s}}{s!(\lambda-s)!(p-s)!(q-\lambda+s)!}, \tag{24}$$

and

$$B_p^q(\gamma, s, t) := -G_{st}(\gamma, p, q) + \delta_{st} \sum_{r=1}^{3} G_{rr}(\gamma, p, q), \tag{25}$$

where

$$G_{st}(\gamma, p, q) = \int_{\boldsymbol{g}\in\mathbb{R}^3} |\boldsymbol{g}|^\gamma g_s g_t H^p(\boldsymbol{g}) H^q(\boldsymbol{g}) \mathcal{M}(\boldsymbol{g}) \,\mathrm{d}\boldsymbol{g}, \qquad s, t = 1, 2, 3. \tag{26}$$

Singularity

We are left with the task to simplify $G_{st}(\gamma, p, q)$. Permutation symmetry would require us only to compute for $s = t = 1$, and $s = 1, t = 3$.

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1. When $\gamma > -3$, it can be computed directly by the recursive formula of the Hermite Polynomials.
2. For the Coulombian case $\gamma = -3$, the recursive formula can not be adopted directly due to the singularity induced by the small value of γ.
3. We will introduce the Burnett Polynomials to deal with the super singularity for $\gamma > -5$.

The normalized form of the Burnett polynomials is

$$B_{\hat{\alpha}}(\boldsymbol{v}) = \sqrt{\frac{2^{1-\hat{\alpha}_1}\pi^{3/2}\hat{\alpha}_3!}{\Gamma(\hat{\alpha}_3+\hat{\alpha}_1+3/2)}} L_{\hat{\alpha}_3}^{(\hat{\alpha}_1+1/2)}\left(\frac{|\boldsymbol{v}|^2}{2}\right)|\boldsymbol{v}|^{\hat{\alpha}_1} Y_{\hat{\alpha}_1}^{\hat{\alpha}_2}\left(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\right),$$

Hermite polynomials in is expressed by a linear combination of the Burnett polynomials, precisely

$$H^{\alpha}(\boldsymbol{v}) = \sum_{|\hat{\alpha}|_B=|\alpha|} C_{\hat{\alpha}}^{\alpha} B_{\hat{\alpha}}(\boldsymbol{v}), \qquad C_{\hat{\alpha}}^{\alpha} = \int_{\mathbb{R}^3} B_{\hat{\alpha}}(\boldsymbol{v}) H^{\alpha}(\boldsymbol{v}) \mathcal{M}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{v}, \quad (27)$$

where $|\hat{\alpha}|_B = \hat{\alpha}_1 + 2\hat{\alpha}_3$. The coefficients could be derived explicitly by the orthogonality of Burnett Polynomials.

Theorem

When $\gamma > -5$*,* $G_{st}(\gamma, p, q)$ *defined in* (26) *can be simplified as*

$$G_{st}(\gamma, p, q) = 2^{(\gamma+2)/2} \sum_{|\hat{p}|_B = |p|} \sum_{|\hat{q}|_B = |q|} C_{\hat{p}}^{p} C_{\hat{q}}^{\hat{q}} D_{\hat{p}_3, \hat{q}_3}^{\hat{p}_1 \hat{q}_1} K\left(\frac{\gamma + \hat{p}_1 + \hat{q}_1 + 3}{2}, \hat{p}_1 + \frac{1}{2}, \hat{q}_1 + \frac{1}{2}, \hat{p}_3, \hat{q}_3\right) F_{st}(\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2), \tag{28}$$

where

$$F_{st}(\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2) = \int_{\mathbb{S}^2} n_s n_t Y_{\hat{p}_1}^{\hat{p}_2}(\boldsymbol{n}) Y_{\hat{q}_1}^{\hat{q}_2}(\boldsymbol{n}) \, \mathrm{d}\boldsymbol{n}, \qquad s, t = 1, 2, 3, \tag{29}$$

The parameters in (28) are defined as

$$D_{n_1 n_2}^{l_1 l_2} = \sqrt{\frac{n_1! n_2!}{\Gamma(n_1 + l_1 + 3/2)\Gamma(n_2 + l_2 + 3/2)}}$$

and

$$K(\mu, \alpha, \kappa, m, n) = (-1)^{m+n}\Gamma(\mu + 1) \sum_{i=0}^{\min(m,n)} \binom{\mu - \alpha}{m - i} \binom{\mu - \kappa}{n - i} \binom{i + \mu}{i}.$$

Finally, with η_{lm}^{μ} defined as

$$\eta_{lm}^{\mu} = \sqrt{\frac{[l + (2\delta_{1,\mu} - 1)m + \delta_{1,\mu}][l - (2\delta_{-1,\mu} - 1)m + \delta_{-1,\mu}]}{2^{|\mu|}(2l-1)(2l+1)}}.$$

$F_{13}(l_1, m_1, l_2, m_2)$ and $F_{33}(l_1, m_1, l_2, m_2)$ equal respectively

$$\frac{(-1)^{m_2+1}}{\sqrt{2}} \sum_{k,j,l=0,1} (-1)^{l+j} \eta^{0}_{\delta_{0k}+(-1)^k l_2, m_2} \eta^{(-1)^l}_{(-1)^j l_1 + \delta_{0j}, m_1} \delta^{m_1+(-1)^l, -m_2}_{l_1+\delta_{1k}-\delta_{1j}, l_2-\delta_{1k}+\delta_{1j}},$$
$$(-1)^{m_2} \sum_{k,j=0,1} \eta^{0}_{\delta_{0k}+(-1)^k l_2, m_2} \eta^{0}_{(-1)^j l_1 + \delta_{0j}, m_1} \delta^{m_1, -m_2}_{l_1+\delta_{1k}-\delta_{1j}, l_2-\delta_{1k}+\delta_{1j}}. \tag{30}$$

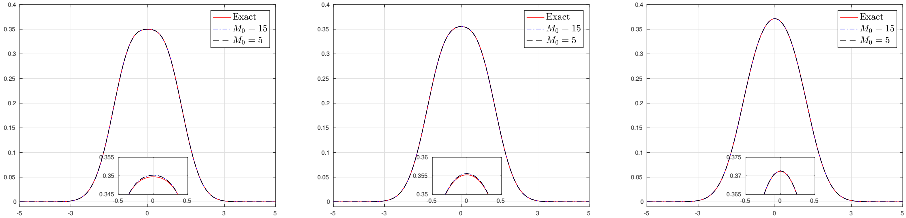
Thus we obtain the explicit expressions.

Section 4

Numerical Examples

BKW

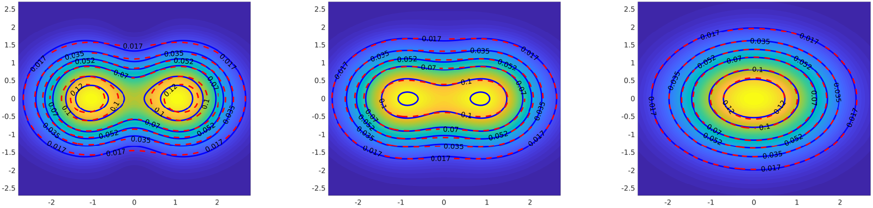
We perform numerical experiments on BKW solution, Bi-gaussian data, and the Rosenbluth Problem. We choose γ to be $0, -3, -4.9$ respectively.



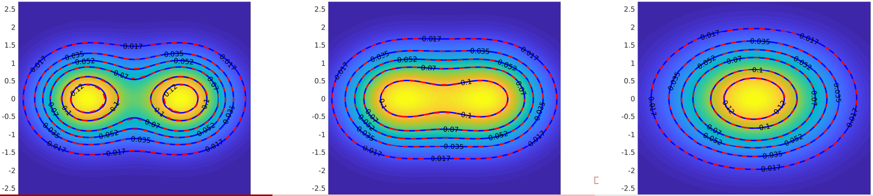
(a) $t = 0.01$ (b) $t = 0.02$ (c) $t = 0.06$

Figure: Marginal distribution functions $g(t, v_1)$ for $M_0 = 5$ and 15 at $t = 0.01$, 0.02 and 0.06. The red solid lines correspond to the exact solution, and the blue dot dashed and black dashed lines correspond to the numerical solutions with $M_0 = 15$ and 5 respectively.

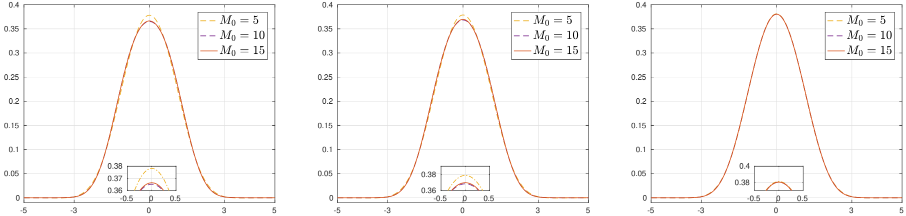
Bi-Gaussian



(a) $t = 0.4, M_0 = 5$ (b) $t = 1, M_0 = 5$ (c) $t = 3, M_0 = 5$



Rosenbluth



(g) $t = 0.4$ (h) $t = 0.6$ (i) $t = 2$

Figure: The Coulombian case $\gamma = -4.9$. Marginal distribution $g(t, v_1)$ functions at different times.

Conclusions

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3. Burnett Polynomials for singular parts in the simplification of exact coefficients.

Future Work

1. Numerical tests for the full FPL equation with spatial variables.
2. Numerical approximation of the collision operator to FPL equation coupled with Maxwell equation, Vlasov-Poisson equation, etc.
3. Boundedness property of the collision operator in the energy norms
4. Convergence and stability of the nonlinear spectral method would follow from the boundedness property.

Section 5

Convergence for Nonlinear Kinetic Equations

Variational Formulation

Exact solution f, truncated space $\mathcal{F}_M$, projection on $\mathcal{F}_M$: f_M, numerical solution $\hat{f}_M$, $u_M = f_M - \hat{f}_M$
Evolution for $\hat{f}_M$:

$$\frac{\partial \hat{f}}{\partial t} = P_M \mathcal{Q}[\hat{f}], \tag{31}$$

Variational formulation:

$$(\frac{\partial u_M}{\partial t}, g) = H(g) + G(g) + (\frac{\partial (f_M - f)}{\partial t}, g) \qquad \forall g \in \mathcal{F}_M$$

where $H(\cdot) = (Q(f_M, f_M) - Q(\hat{f}_M, \hat{f}_M), \cdot)$,
$G(\cdot) = (Q(f, f) - Q(f_M, f_M), \cdot)$.

Nonlinear Evolution Problem

$$(\frac{\partial u_M}{\partial t}, g) = H(g) + G(g) \qquad \forall g \in \mathcal{F}_M$$

Insert $g = u_M$, then

$$\frac{1}{2}\frac{\partial}{\partial t}||u_M||^2 = H(u_M) + G(u_M) \tag{32}$$

Spectral Accuracy of the Projection

If $f \in H^r(\mathbb{R}^N; M^{-1}dv)$, there exists a constant c independent of f, such that

$$||f_M - f|| \leq cM^{-Nr/2}||f||_r$$

Convergence for Bounded Kernel

If the collision kernel has the following boundedness of the norm (for example, Boltzmann collision operator with compactly supported kernel; linear Boltzmann operator):

$$||Q(f,g)|| \leq ||f||||g|| \tag{33}$$

$$|H(u_M)| \leq ||u_M||||Q(f_M - \hat{f_M}, f_M) + Q(\hat{f}_M, f_M - \hat{f}_M)|| \tag{34}$$

$$\leq C||u_M||||u_M||(2||f|| + ||u_M||) \tag{35}$$

Similarly,

$$|G(u_M)| \leq ||u_M||||Q(f - f_M, f) + Q(f_M, f - f_M)|| \tag{36}$$

$$\leq C||u_M||||f - f_M||(2||f||) \tag{37}$$

Convergence for Bounded Kernel

$$|\frac{\partial}{\partial t}||u_M||| \leq 2C||u_M||||f|| + 2C||u_M||^2 + 2cC||f||||f||_r M^{-Nr/2} \quad (38)$$

$$|\frac{\partial}{\partial t}||u_M||| \leq C_f||u_M|| + c_f M^{-Nr/2} \quad (39)$$

where $C_f = 2C||||f||||_{L^\infty} + 2C$, $c_f = 2cC||||f||||_{L^\infty}||||f||_r||_{L^\infty}$

$$||u_M|| \leq \exp(C_f t)(\delta + \frac{c_f M^{-Nr/2}}{C_f})$$

Convergence for Bounded Kernel

The following cases can be categorized to have the stated boundedness property of the norm:

1. Collisional Kernel with compact support,
2. Linearized Collisional Operator.

General Case: Potential ideas

In general, the boundedness of the collision kernel is false. (Boltzmann kernel with unbounded radial part)
The residual part G: proceed as previously.
Collision part H: $(Q(f_M, f_M) - Q(\hat{f}_M, \hat{f}_M), u_M)$.
In general, we can get $||Q(f,g)|| \leq ||f||_\alpha ||g||_\alpha$. We need to invoke a certain coercivity to compensate the increase of the index α.

General Case: Coercivity of the linearized collision kernel

One way is to put $Q = Q_{linear} + Q_{res}$.

$$\int_{\mathbf{R}^3} \phi \mathcal{L}_M \phi M dv \geq c_0 \int_{\mathbf{R}^3} \phi^2 \lambda(|v|) M dv \tag{40}$$

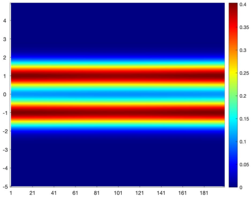
$$\frac{1}{c_1}(1 + |v|^\gamma) \leq \lambda(|v|) \leq c_1(1 + |v|\gamma) \tag{41}$$

for Boltzmann hard potential with $\gamma \leq 1$. Where $L_M = -Q_{linear}$
Therefore, we only need to show
$(Q_{res}(f_M) - Q_{res}(\hat{f}_M), u_M) \leq \delta ||u_M||_\gamma + C(\delta)||u_M||$.

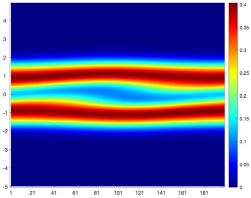
Section 6

Numerical Illustrations with inhomogeneous equation

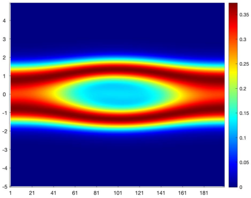
Two Stream



(a) $t = 0$

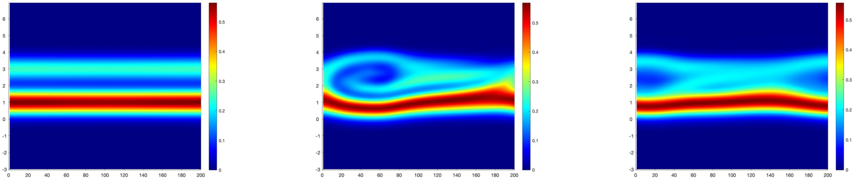


(b) $t = 0.2$



(c) $t = 0.5$

Bump-on-tail



(d) $t = 0$ (e) $t = 0.4$ (f) $t = 0.7$

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Thanks!