Stable type-I blowup by local normalization conditions: nonlinear heat and complex Ginzburg-Landau equations

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1 Type-I blowup with log correction

2 Semilinear heat equation

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4 Discussions and future work

Millennium prize problem: blowup of 3D NS equation

- 3D incompressible Navier-Stokes equation:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0\,. \tag{1}$$

- Euler equation: $\nu = 0$. NS equation $\nu > 0$.
- Blowup of a quantity of interest f:

$$\limsup_{t \to T^-} \|f(t)\|_{L^\infty} = \infty, \quad T < +\infty\,.$$

- Millennium prize problem: global well-posedness or finite time blowup of (1) from **smooth** initial data **on the whole space**.

Self-similar blowup

- **Structured** singularity with less DoF and **self-similar** blowup:

$$f(t, \mathbf{x}) = (T - t)^{c_f} F(\mathbf{x}/(T - t)^{c_l}) \,. \tag{2}$$

  T: blowup time; $c_f < 0$: blowup rate.

- Hou-Luo 2013: numerical evidence of self-similar blowup for **smooth** initial data of 3D axisymmetric Euler equation with **boundary**. Elgindi, Chen-Hou-Huang,... **Chen-Hou 2022**: rigorous proof.

- Hou-W. 2024: Self-similar singularity of a modified 1D Hou-Li (2008) model mimicking the interior blowup of Euler/NSE.

- Original 3D Euler/NSE might not have self-similar singularity under certain assumptions: Chae 2007; Tsai 1998; Hou 2024.

Nonlinear heat and complex Ginzburg-Landau equations

$$\psi_t = (1+\imath\beta)\Delta\psi + (1+\imath\delta)|\psi|^{p-1}\psi - \gamma\psi, \tag{CGL}$$

- Reduces to nonlinear heat equation (NLH) when $\beta = \delta = \gamma = 0$.
- Connects to nonlinear Schrödinger equation (NLS) as $\beta, |\delta| \to \infty$.
- Blowup asymptotics in subcritical range $\flat_* := p - \delta^2 - \beta\delta(p+1) > 0$:

$$\psi(x,t) \sim |\log(T-t)|^{\imath\mu}\Big[(T-t)\big(p-1+c_p|Z|^2\big)\Big]^{-\frac{1+\imath\delta}{p-1}},$$
$$Z = \frac{x}{\sqrt{(T-t)|\log(T-t)|}}, \quad c_p = \frac{(p-1)^2}{4\flat_*}, \quad \mu = -\frac{\beta(1+\delta^2)}{2\flat_*}. \tag{3}$$

Existing literature and goal

- Existing literature relies on explicit profile and spectrum analysis. Stability is stated for well-prepared data and topological argument.
  - NLH: numerics using dynamic rescaling, Berger-Kohn 1988;
  - NLH: stability of blowup, Bricmont-Kupiainen 1994, Merle-Zaag 1997;
  - CGL: stability of blowup via spectral analysis, Masmoudi-Zaag 2008.

- Can we develop a framework for Type-I blowup
  - with clear characterization of stability;
  - aligning well with numerics for numerical profile discovery;
  - amenable to computer-assisted proofs in the case without spectrum or even any explicit profile?

NLH: Hou-Nguyen-W. 2024, CGL: Chen-Hou-Nguyen-W. 2024

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Dynamic rescaling formulation in 1D

- Semilinear heat equation:

$$\psi_t = \Delta\psi + \psi^2. \tag{SLH}$$

- Dynamic rescaling formulation:

$$U(z,\tau) = H(\tau)\psi(D(\tau)z, t(\tau))\,, U_\tau = c_U U - d_U z U_z + U^2 + \frac{H}{D^2}U_{zz}\,.$$
$$H = H(0)\exp\Big(\int_0^\tau c_U d\tau\Big)\,, D = \exp\Big(\int_0^\tau -d_U d\tau\Big)\,, t = \int_0^\tau H d\tau\,. \tag{4}$$

- Approximate profile:

$$\bar{U} = (1+z^2/8)^{-1}\,, c_{\bar{U}}\bar{U} - d_{\bar{U}} z \bar{U}_z + \bar{U}^2 = 0\,, c_{\bar{U}} = -1\,, d_{\bar{U}} = 1/2\,.$$

To the leading order, rescaled SLH is just rescaled Riccati equation.

Stability via local vanishing conditions

- Perturbative ansatz:

$$U = \bar{U} + W \, , c_U = c_{\bar{U}} + c_W \, , d_U = d_{\bar{U}} + d_W \, . \tag{5}$$

- Modulation via local vanishing conditions: enforcing W even and $W(0) = W_{zz}(0) = 0$.
- ODE of modulation conditions:

$$\lambda := \frac{H}{D^2} = H(0) \exp\big(\int_0^\tau c_W + 2 d_W d\tau \big) \, ,$$

$$c_W = \frac{1}{4}\lambda \, , d_W = -(\frac{5}{8} + 2 W_{zzzz}(0))\lambda \, , \lambda_\tau = -(1 + 4 W_{zzzz}(0))\lambda^2 \, . \tag{6}$$

$\lambda \approx 1/\tau \approx \frac{1}{\log|T-t|}$. Viscosity terms are treated perturbatively.

Local vanishing conditions explained

- Motivation: Ricatti equation.
- For a weight $\rho = z^{-\alpha}$ using L^2 estimate, near the origin

$$(W_\tau, W\rho) \approx (-1 + \frac{1}{4}\frac{(\rho z)_z}{\rho} + 2)(W, W\rho) = (1 - \frac{\alpha - 1}{4})(W, W\rho)\,.$$

- Singular weights

$$\rho_0 = z^{-6} + 10^{-3}\,, \rho_k = \rho_0 z^{2k} \text{ if } k \leq 3\,, \rho_k = 1 + 100^{-k} z^{2k} \text{ if } k > 3\,.$$

- Energy estimate for small μ and $E^2 = \sum_{k=0}^{5} \mu^k E_k^2$:

$$\partial_\tau E \leq (-\frac{1}{10} + CE\lambda + CE)E + C\lambda + C\lambda E\,. \tag{7}$$

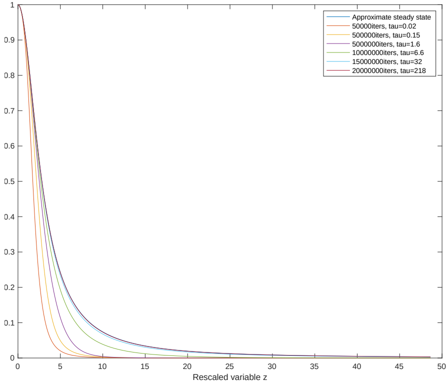
Choosing small $\lambda(0)$, we have stability and the law of blowup.

Numerical result

Starting from initial value beyond our smallness assumption

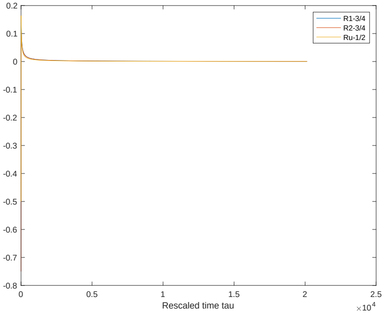
$$U(0,z) = (1 + z^2/8 + z^4/10)^{-1}\,, H(0) = 1\,,$$

and using vanishing modulation condition numerically, profile converges.



Generalization to higher dimensions

- Introduce d-different rescalings in $z_1, z_2, \cdots, z_d$ and impose even symmetries. We have $1+d$ vanishing conditions which matches the number of modulation parameters.
- Nonradial initial condition in 2D. Normalization constants converge to the correct rate numerically: $R_i := d_W^i \tau \to 3/4$, $R_u := c_W \tau \to 1/2$.



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Generalized dynamic rescaling formulation in nD

- Amplitude-phase representation: $\psi(x,t) = u(x,t)e^{\imath\theta(x,t)}$.
- Full stability w/o even symmetry assumption, $1 + d + d(d+1)/2$ DoF:

$$U(z,\tau) = H(\tau)u\big(\mathbf{R}(\tau)z + V(\tau), t(\tau)\big)\,,$$

$$\Theta(z,\tau) = \theta\big(\mathbf{R}(\tau)z + V(\tau), t(\tau)\big), \quad t(\tau) = \int_0^\tau H^{p-1}(s)ds,$$

where $\mathbf{R}(\tau) \in \mathbb{R}^{d\times d}$ upper triangular, $V(\tau) \in \mathbb{R}^d$ and $H(\tau) \in \mathbb{R}_+$.
- The modulation corresponds to the symmetries of the equation, with $d-1$ extra modulation parameters.

Modulation equation and linearization

- Linearization around the approximate steady state:

$$U = \bar{U} + W\,, \Theta = \bar{\Theta} + \Phi\,, c_U = -\frac{1}{p-1} + c_W\,, H = e^{-\frac{\tau}{p-1}} C_W\,.$$

- Modulation equation:

$$\mathcal{M}^{-1} = e^{-\tau/2}\mathbf{R}, \mathcal{V} = -\mathbf{R}^{-1}\dot{V}, \mathcal{P} = \dot{\mathcal{M}}\mathcal{M}^{-1}, \mathcal{Q} := C_W^{p-1}\mathcal{M}\mathcal{M}^T\,.$$

$$U_\tau = c_U U - (\frac{1}{2}z + \mathcal{P}z + \mathcal{V}) \cdot \nabla U + U^p - C_U^{p-1}\gamma U + \mathscr{D}_U\,,$$

$$\Theta_\tau = -(\frac{1}{2}z + \mathcal{P}z + \mathcal{V}) \cdot \nabla \Theta + \delta U^{p-1} + \mathscr{D}_\Theta\,.$$

Diffusion terms are of order $\mathcal{Q}$.

Law of the singularity via local vanishing conditions

- Enforcing $W = O(|z|^3)$ at the origin, we get an ODE of modulation parameters. In particular

$$\mathrm{tr}(\mathcal{Q})_\tau \approx -\mathrm{tr}(\mathcal{Q}^2)\,, \quad \mathrm{tr}(\mathcal{Q}^{-1})_\tau \approx -d\,.$$

- Trace estimate gives $\mathcal{Q} \approx \tau^{-1}\mathsf{I}_d$.

  $\mathcal{Q}$ may be anisotrophic initially, but would end up isotropic.

Technical challenges of stability analysis

- General nonlinearity and the phase equation necessitate a lower bound of U: maximal principle and weighted L^∞ estimate.
- Sharp decay estimates of $\nabla^i U$: almost tight power for the weights and interpolation, embedding inequalities.
- Coupling of amplitude and phase: top-order energy with special algebraic structure to cancel out top-order terms in diffusion.

$$\mathscr{D}_U = \Delta_{\mathcal{Q}} U - 2\beta \langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}} - U \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}} - \beta U \Delta_{\mathcal{Q}} \Theta \,,$$
$$\mathscr{D}_\Theta = \beta \frac{\Delta_{\mathcal{Q}} U}{U} + 2 \frac{\langle \nabla U, \nabla \Theta \rangle_{\mathcal{Q}}}{U} - \beta \langle \nabla \Theta, \nabla \Theta \rangle_{\mathcal{Q}} + \Delta_{\mathcal{Q}} \Theta \,.$$

We construct top-order energy as

$$(|\nabla^k W|^2, \rho_k) + (|\nabla^k \Phi|^2, U^2 \rho_k) \,.$$

Discussions and future work

- Our method is robust, with clear stability and no need for spectral analysis. One can hope to combine with computer-assisted proofs.
- For type I blowup, local vanishing modulation conditions corresponds to local orthogonality conditions for the unstable eigenmodes of the linearized operator.
- How to generalize this approach to type II blowup?