Self-similar blowup for a 1D quasi-exact model of Navier-Stokes

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Millennium prize problem: blowup of 3D NS equation

- 3D incompressible Navier-Stokes equation:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{p} + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0\,. \tag{1}$$

- Euler equation: $\nu = 0$. NS equation $\nu > 0$.

- Blowup of a quantity of interest f:

$$\limsup_{t \to T^-} \|f(t)\|_{L^\infty} = \infty, \quad T < +\infty\,.$$

- Millennium prize problem: global well-posedness or finite time blowup of (1) from **smooth** initial data **on the whole space**.

Self-similar blowup and axisymmetric equation

- **Structured** singularity with less DoF
  - **Self-similar** blowup:

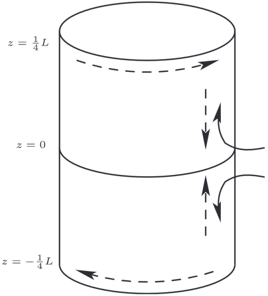
$$f(t, \mathbf{x}) = (T-t)^{c_f} F(\mathbf{x}/(T-t)^{c_l}) \,. \tag{2}$$

    T: blowup time; $c_f < 0$: blowup rate.
  - **Axisymmetric** Euler equation: cylindrical formulation (r, z, θ), velocity independent of θ.

- Hou-Luo 2013 : numerical evidence of self-similar blowup for **smooth** initial data of 3D axisymmetric Euler equation with **boundary**. Chen-Hou 2022 : rigorous proof of blowup.

Self-similar blowup candidates of 3D axisymmetric Euler

- Blowup on the **boundary** for H-L case



Boundary helps blowup!

- **Our goal**: identify and understand blowup in the **interior**
  - (equivalently) Approaching millennium prize problem.
  - Generalize blowup mechanism from 1D to 3D.

1D models with self-similar blowup: gCLM

- Vorticity formulation $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ for 3D Euler:

$$\boldsymbol{\omega}_t + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \nabla \mathbf{u} \cdot \boldsymbol{\omega}\,. \tag{3}$$

- Biot-Savart law, $\boldsymbol{\omega} \to \mathbf{u}$ via nonlocal interaction: $\nabla \mathbf{u} = \mathcal{R}(\boldsymbol{\omega})$.
  $\mathcal{R}$: Riesz transform.

- 1D model: generalized CLM model (Okamoto-Sakajo-Wunsch 2008):

$$\omega_t + a u \omega_x = u_x \omega, \quad u_x = H\omega\,. \tag{4}$$

  H: Hilbert transform, 1D analogue of Riesz transform.
  a: strength of **advection** in competition with **vortex stretching**.

Dynamic rescaling formulation

- **Profile equation** for 1d gCLM model $\omega_t + au\omega_x = u_x\omega$. Plugging in

  $$\omega(t,x) = (T-t)^{c_\omega}\Omega(x/(T-t)^{c_l})\,, u(t,x) = (T-t)^{c_\omega+c_l}U(x/(T-t)^{c_l})$$

  the self-similar ansatz and balance the terms in t, we get:

  $$(c_l y + aU)\,\Omega_y = (c_\omega + U_y)\,\Omega, \quad U_y = H\Omega\,. \tag{5}$$

- **Dynamic rescaling formulation** (DRF) for time-depedent c_l, c_ω:

  $$\Omega_\tau + (c_l y + aU)\,\Omega_y = (c_\omega + U_y)\,\Omega, \quad U_x = H\Omega\,. \tag{6}$$

  Equivalent to original equation by time rescaling.
  Steady state recovers profiles.
  Finite time blowup holds if $c_\omega \le -C < 0$ and is self-similar if Ω, U converge to a profile.

A more faithful 1D model: Hou-Li

- N-S equation in axisymmetric case:

$$\begin{aligned} u_{1,t} - r\psi_{1,z}u_{1,r} + (2\psi_1 + r\psi_{1,r})u_{1,z} &= 2u_1\psi_{1,z} + \nu\Delta u_1 \,, \\ \omega_{1,t} - r\psi_{1,z}\omega_{1,r} + (2\psi_1 + r\psi_{1,r})\omega_{1,z} &= \left(u_1^2\right)_z + \nu\Delta\omega_1 \,, \\ -\left[\partial_r^2 + (3/r)\partial_r + \partial_z^2\right]\psi_1 &= \omega_1 \,. \end{aligned} \tag{7}$$

- Hou-Li (2008) constant approximation in r-direction:

$$\begin{aligned} u_t + 2\psi u_z &= 2u\psi_z + \nu u_{zz} \,, \\ \omega_t + 2\psi\omega_z &= \left(u^2\right)_z + \nu\omega_{zz} \,, \\ -\psi_{zz} &= \omega \,. \end{aligned} \tag{8}$$

- Model is well-posed in C^1; convection is weaker in 3D.

Our result on Hou-Li model

Weak convection model:

$$\begin{aligned} u_t + 2a\psi u_x &= 2u\psi_x + \nu u_{xx} \,, \\ \omega_t + 2a\psi\omega_x &= \left(u^2\right)_x + \nu\omega_{xx} \,, \\ -\psi_{xx} &= \omega \,. \end{aligned} \tag{9}$$

Our work: (forthcoming paper with Hou)

Theorem (Blowup of (9) with periodicity in x)

There exists steady blowups with scaling index in space $c_l = 0$, for

1. $a < 1$ *close to* 1, $\nu = 0$, *self-similar blowup with smooth data;*
2. $a < 1$ *close to* 1, $\nu > 0$, *blowup with smooth data;*
3. $a = 1$, $\nu = 0$, *self-similar blowup with any Hölder* $\alpha < 1$ *regularity.*

The Hou-Li model shares many similarities to De Gregorio model.

Proof of linear stability

- Explicit approximate profiles: from the steady state for $a = 1$.

$$(\bar{\omega}, \bar{u}, \bar{\psi}) = (\sin x, \sin x, \sin x).$$

- Linear stability for the perturbation:

$$D := \frac{1}{2}\frac{d}{dt}(\|u\|_{\chi_1}^2 + \|\omega\|_{\chi_2}^2) \approx (L_1, u)_{\chi_1} + (L_2, \omega)_{\chi_2} \lesssim -[\|u\|_{\chi_1}^2 + \|\omega\|_{\chi_2}^2] \,.$$

$$L_1 = -2\sin x u_x - 2\cos x\psi + 2u\cos x + 2\sin x\psi_x \,,$$

$$L_2 = -2\sin x\omega_x - 2\cos x\psi + 2u\cos x + 2\sin x u_x \,.$$

- Singular weights: $\rho_0 = \frac{1}{1-\cos x}, \rho_k = (1+\cos x)^k$ with the norm

$$E_k^2(t) = (u^{(k+1)}, u^{(k+1)}\rho_k) + (\omega^{(k)}, \omega^{(k)}\rho_k) \,.$$

Damping in the leading order term.

Difficulties in linear estimate

- Estimate of local and nonlocal terms in L^2:

$$\begin{aligned} D_0 = &-\left[(u_x, u_x\rho) + (\omega, \omega\rho) + (u, u\rho)\right] \\ &+ 2\left[-(\cos x\psi, \omega\rho) + (\sin x\psi, u_x\rho) + (u\cos x, \omega\rho)\right]. \end{aligned}$$

- **Exact** computation in Fourier basis to avoid overestimate.
- Establish negative-definiteness of **quadratic form** (w.r.t Fourier coefficients) with decaying entries.
- **Computer-assisted verification** of finite truncation: the quadratic form projected in first 200 basis.

Backup slide: difficulties in viscous terms

- H^k estimates for the viscous term spit out for example

$$(\omega^{(k+2)}, \omega^{(k)} \rho_k) \approx -(\omega^{(k+1)}, \omega^{(k+1)} \rho_k) + C(k)(\omega^{(k)}, \omega^{(k)} \rho_{k-1}) \,.$$

- Criteria for the norm:
  - Linear damping
  - Stronger than $W^{3,\infty}$-norm near the origin
  - Control on viscous terms

- Combination of a cascade of norms to close the estimate:

$$I^2 = \sum_{k=0}^{4} E_k^2 \mu^k \,, \quad \mu \ll 1 \,.$$