Exponentially Convergent Multiscale Finite Element Method

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May 23, 2023

Summary of our contribution: ExpMsFEM

Systematic approach for solving multi-query multiscale problems efficiently using **offline bases**, with state-of-the-art accuracy **rigorously**.

- For elliptic equations: *Multiscale Modeling & Simulation* 2021
- For Helmholtz equations: *Multiscale Modeling & Simulation* 2023
- Review paper: *Communications on Applied Mathematics and Computation* 2023

Joint work with Chen, Hou.
Ongoing collaboration on generalization to the Schrödinger equation.

Multiscale model reduction

- **Model problem** in 2D and 3D:

  $$-\nabla \cdot (A(x)\nabla u) - P(x)u = f, \text{ in } \Omega \subset \mathbb{R}^d, \quad \text{w/ boundary conditions}$$

  wave mechanics, subsurface flows, electrostatics, seismology.
  - Heterogeneity: $A, P \in L^\infty(\Omega)$ without scale separation. $0 < A_{\min} \leq A(x) \leq A_{\max}$. $f \in L^2(\Omega)$.
  - Highly **Oscillatory** solutions.

- **Model reduction**: use a small number of local basis functions to achieve desired **accuracy theoretically and numerically**.
  - Desirable if same offline bases can be used with different f.

Literature on multiscale methods for elliptic equations

- **Local** bases + **global** coupling
  - Multiscale Finite Element Methods (MsFEM): Hou, Wu 1997
  - Genealized Finite Element Methods (GFEM) via Partition of Unity Method (**PUM**): Babuska, Lipton 2011
- **Global** bases via variational problem + **local** truncation
  - Gamblets: Owhadi-Zhang-Berlyand 2014
  - Localizable Orthogonal Decompositions (**LOD**): Malqvist, Peterseim 2014
- VMS 1998, HMM 2003...

Helmholtz equation and pollution effect

Helmholtz equation with high wave number k:

$$\mathcal{L}_k u := -\nabla \cdot (A\nabla u) - k^2 V^2 u = f, \text{ in } \Omega, \quad \text{w/ boundary conditions}$$

where $V \in L^\infty(\Omega)$.

- Numerical difficulty: **pollution effect** (Babuska, Sauter 1997)
  - Maximal mesh size to address the wave length: $O(1/k)$.
  - Standard FEM: local mesh size $H = O(1/k^2)$.
  - Ideal method: $H = O(1/k)$!
- Mathematical challenge: **indefinite** operator.

Overcoming the pollution effect

Two key insights and methods that capture oscillation with $O(H)$ error

- **Gårding-type inequality**: good approximation implies good solution. $hp-$FEM with polynomial of order $O(\log k)$. (Melenk, Sauter 2010)

- **Poincaré inequality**: local problem resembles elliptic problem. LOD with support size $O(H\log(1/H)\log k)$. (Peterseim 2017)

Our method: Best of (G) and (P)

- ExpMsFEM with first exponential rate of convergence. (C-H-W)

- Later: PUM with same rate of convergence. (Ma-Alber-Scheichl)

Four methods have comparable complexity if aimed at minimal accuracy: $O(1/k)$ error in **energy norm**, mesh size $O(1/k)$, DoF $O(k^d \text{poly}(\log(k)))$.

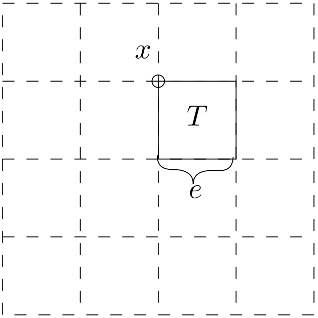
Explore the solution space (G)

- **Mesh structure** in 2D:
  nodes, edges and elements.

- **Split** the solution locally (P):
  in each T, $u = u_T^{\mathsf{h}} + u_T^{\mathsf{b}}$.

$$\begin{cases} \mathcal{L}_k u_T^{\mathsf{h}} = 0 \ \text{ in } T \\ u_T^{\mathsf{h}} = u \ \text{ on } \partial T, \end{cases}$$
$$\begin{cases} \mathcal{L}_k u_T^{\mathsf{b}} = f \ \text{ in } T \\ u_T^{\mathsf{b}} = 0 \ \text{ on } \partial T. \end{cases}$$

- **Merge**: For each T, $u^{\mathsf{h}}(x) = u_T^{\mathsf{h}}(x)$ and $u^{\mathsf{b}}(x) = u_T^{\mathsf{b}}(x)$, when $x \in T$.



$x \in \mathcal{N}_H, e \in \mathcal{E}_H, T \in \mathcal{T}_H$

Key insights of exponential accuracy

- (Generalized) **harmonic-bubble splitting** (Hetmaniuk, Lehoucq 2010), (Hou, Liu 2016)
- **Edge localization**
- **Oversampling** (Hou, Wu 1997) for low-complexity edge space

Theorem (Informal statement of exponentially efficient edge bases)

Suppose $H = O(1/k)$, then for each edge e, we can find m local edge bases such that the relative error using those edge bases to approximate any edge function is at most $C \exp\left(-bm^{\frac{1}{d+1}}\right)$.

Sketch of our result

On a mesh of lengthscale $H = O(1/k)$, u can be computed by

$$u = \underbrace{\sum_{i \in I_1} c_i^f \psi_i^{(1)}}_{\text{(I)}} + \underbrace{\sum_{i \in I_2} \psi_i^f}_{\text{(II)},O(H)} + C \exp(-bm^{\frac{1}{d+1}}) \qquad \text{(Energy norm)}$$

b, C constants independent of H, k. $\psi_i^{(1)}, \psi_i^f$ *local* support of size H.

- $\psi_i^{(1)}$ via *local* SVD of $\mathcal{L}_k$, offline, parallelizable $\#I_1 = O(m/H^d)$
- ψ_i^f via solving *locally* $\mathcal{L}_k u = f$ online, parallelizable $\#I_2 = O(1/H^d)$
- c_i^f obtained by Galerkin methods with bases $\psi_i^{(1)}$; offline matrix

A data-adaptive coarse-fine scale decomposition

Artificial example with rough media and high wavelength

Rough media, high wavelength $k = 2^5$ with mixed boundary conditions.

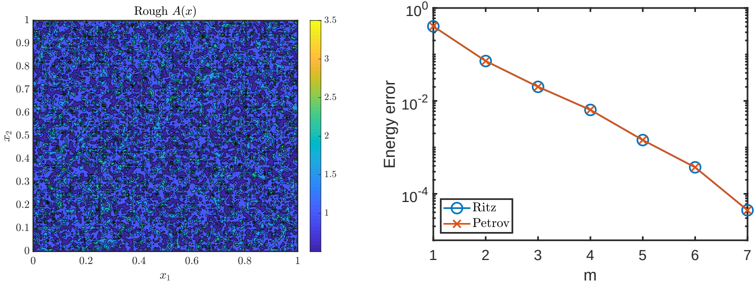


Figure: Left: the contour of A; right: relative errors in the energy norm.

Exponential decaying error; works better in practice than PUM.

Backup example of high wavenumber

- $A = V = \beta = 1$, $k = 2^7$, fine mesh $h = 2^{-10}$, coarse mesh $H = 2^{-5}$.
- Exact solution: $u(x_1, x_2) = \exp(-ik(0.6x_1 + 0.8x_2))$.

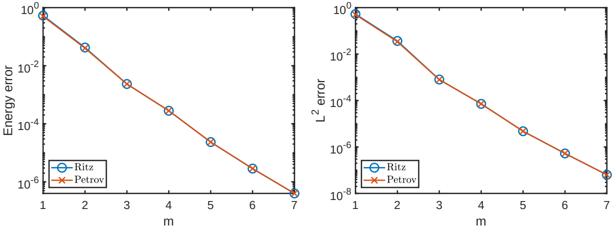


Figure: High wavenumber example. Left: $e_{\mathcal{H}}$ versus m; right: e_{L^2} versus m.

Backup example of high contrast: Mie resonances

$$\Omega_\varepsilon = (0.25, 0.75)^2 \cap \bigcup_{j\in\mathbb{Z}^2} \varepsilon\left(j + (0.25, 0.75)^2\right), \quad A(x) = \begin{cases} 1, & x \notin \Omega_\varepsilon \\ \varepsilon^2, & x \in \Omega_\varepsilon\,. \end{cases}$$

$\beta = 1, V = 1, k = 9.$

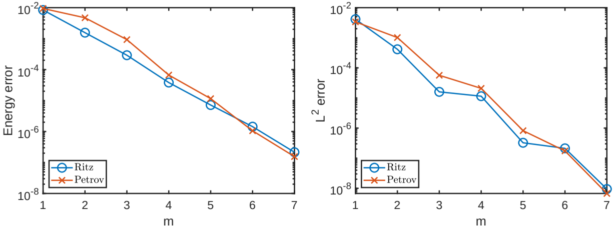


Figure: High contrast example. Left: $e_{\mathcal{H}}$ versus m; right: e_{L^2} versus m.